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By Henry Taber.

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§ 1.

IN what follows $X_1, X_2, \ldots X_r$, will denote r differential operators defined thus:

$$X_j \equiv \sum_1^n{}_i\, \xi_{ji} \; (x_1, x_2, \ldots x_n) \frac{\partial}{\partial x_i}$$
$$(j = 1, 2, \ldots r),$$

where the ξ's are analytic functions of n independent variables x. It will be assumed that the X's are *independent*, that is to say, that no system of quantities $\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_r$, independent of the x's and not all zero, can be found for which

$$(\mathrm{a}_1 X_1 + \mathrm{a}_2 X_2 + \ldots + \mathrm{a}_r X_r) f \equiv \mathrm{a}_1 X_1 f + \mathrm{a}_2 X_2 f + \ldots + \mathrm{a}_r X_r f \equiv 0,$$

for all functions f of the x's; that is, for which

$$\mathrm{a}_1 \xi_{1i} + \mathrm{a}_2 \xi_{2i} + \ldots + \mathrm{a}_r \xi_{ri} \equiv 0,$$

simultaneously, for $i = 1, 2, \ldots n$. By means of these different operators we may construct a family with ∞^r of transformations

(1) $$x'_i = f_i (x_1, \ldots x_n, a_1, \ldots a_n)$$
$$(i = 1, 2, \ldots n),$$

where the a's are arbitrary parameters, and $f_i (x, a)$ is defined for values of the a's sufficiently small by the series

$$x_i + \sum_1^r{}_j\, a_j X_j x_i + \tfrac{1}{2} \sum_1^r{}_j \sum_1^r{}_k\, a_j a_k X_j X_k + \text{etc.*}$$

For assigned values of the a's the transformation defined by these equations may be denoted by T_a.

* Lie: Transformationsgruppen, I. pp. 61, 62.

Among the transformations of this family is an ∞^{r-1} of infinitesimal transformations (that is, of transformations infinitely near the identical transformation), obtained by making the a's infinitesimal. Thus let

$$a_1 = \alpha_1 \delta t, a_2 = \alpha_2 \delta t, \ldots a_r = \alpha_r \delta t,$$

where the α's are arbitrary finite quantities independent of the x's, and δt is an infinitesimal constant. The system of equations defining this ∞^{r-1} of infinitesimal transformations is then

$$(2) \qquad x'_i = f_i(x_1, \ldots x_n, \alpha_1 \delta t, \ldots \alpha_r \delta t) = x_i + \delta t \sum_1^r{}_j \alpha_j X_j . x_i$$
$$(i = 1, 2, \ldots n).$$

For assigned values of the α's, the continued applications to the manifold $(x_1, x_2, \ldots x_n)$ of the infinitesimal transformations

$$x_i + \delta t \sum_1^r{}_j \alpha_j X_j . x_i,$$

of which $\sum_1^r{}_j \alpha_j X_j$ is said to be the symbol, generates a group $G_1^{(\alpha)}$ with a single parameter t of transformations

$$(3) \qquad x'_i = f_i(x_1, \ldots x_n, t\alpha_1, \ldots t\alpha_n)$$
$$(i = 1, 2, \ldots n).$$

Thus, if

$$(4) \qquad x''_i = f_i(x'_1, \ldots x'_n, t'\alpha_1, \ldots t'\alpha_n)$$
$$(i = 1, 2, \ldots n),$$

we derive by the elimination of the x''s

$$(5) \qquad x''_i = f_i(x_1, \ldots x_n, t''\alpha_1, \ldots t''\alpha_n)$$
$$(i = 1, 2, \ldots n),$$

where $t'' = t + t'$. In particular, if $t' = -t, x''_i = x_i$ for $i = 1, 2, \ldots n$. Therefore, each transformation of $G_1^{(\alpha)}$ is paired with its inverse and, for $t = 0$, we have the identical transformation.* In accordance with the notation adopted, the general transformation of $G_1^{(\alpha)}$ is denoted by $T_{t\alpha}$; and, by what precedes, if $T_{t\alpha}{}^{-1}$ denotes the transformation inverse to $T_{t\alpha}$, we have $T_{t\alpha}{}^{-1} = T_{-t\alpha}$.

As t approaches infinity the transformation of group $G_1^{(\alpha)}$ defined by (3) may approach a definite finite transformation T. But, although for t infinite, $T_{t\alpha} = \mathrm{T}$ may be non-illusory, it cannot be said to be generated by the infinitesimal transformation of $G_1^{(\alpha)}$. The conception of the

* Lie: Transformationsgruppen, I. pp. 52, 55.

generation of a finite transformation by an infinitesimal transformation is not applicable in this case. Moreover, for $t = \infty$, the resulting transformation of $G_1^{(a)}$ has properly no inverse.

For assigned finite values of the a's, the transformation T_a of the family defined by equations (1), if not illusory, belongs to the group $G_1^{(a)}$ with a single parameter generated by the infinitesimal transformation whose symbol is $\sum_{1}^{r}{}_j\, a_j X_j$. Thus the totality of transformation with finite parameters of the family (1) separate into an ∞^{r-1} of groups $G_1^{(a)}$. In consequence of what has been said relative to the group $G_1^{(a)}$, it follows that each transformation of the family (1) with finite parameters is paired with its inverse, and we have $T_a^{-1} = T_{-a}$.

As the a's approach certain limiting values, of which some are infinite, the transformation T_a may approach a definite finite transformation T as a limit. This transformation may be equivalent to a transformation T_b of the family (1) with finite parameters $b_1, b_2, \ldots b_r$. In this case T is generated by an infinitesimal transformation of the family, namely $\sum_{1}^{r}{}_j\, b_j X_j$, but not otherwise.*

The composition of two arbitrary transformations T_a, T_b of the family, defined, respectively, by the equations

$$(6) \qquad x'_i = f_i(x_1, \ldots x_n, a_1, \ldots a_r) \qquad (i = 1, 2, \ldots n),$$

$$(7) \qquad x''_i = f_i(x'_1, \ldots x'_n, b_1 \ldots b_r) \qquad (i = 1, 2, \ldots n),$$

gives a transformation which may be denoted by $T_b T_a$, and is defined by

$$(8) \qquad x''_i = f_i(f_1(x, a), \ldots f_n(x, a), b_1, \ldots b_r) \qquad (i = 1, 2, \ldots n).$$

This transformation is not, in general, a transformation of the family. It will, however, be assumed throughout this paper that

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* Let $a_1 = \alpha_1 t, a_2 = \alpha_2 t, \ldots a_r = \alpha_r t$. It is obviously necessary to distinguish between the equations of transformation which result from assigning definite finite values to the α's, and then increasing t without limit, and those which result when $a_1, a_2, \ldots a_r$ (without preserving the same ratio) approach severally certain limiting values some of which are infinite. The transformation which results in the first case has properly no inverse. It transforms every point on any one of the path curves of the group $G_1^{(\alpha)}$ into invariant points of such curves. The transformation which results in the second case if non-illusory may possess an inverse.

$$X_j X_k - X_k X_j \equiv \sum_1^r {}_l c_{jkl} X_l$$
$$(j, k = 1, 2, \ldots r),$$

the c's being quantities independent of the x's. In which case, from

(6) $$x'_i = f_i (x_1, \ldots x_n, a_1, \ldots a_r)$$
$$(i = 1, 2, \ldots n),$$

(7) $$x''_i = f_i (x'_1, \ldots x'_n, b_1, \ldots b_r)$$
$$(i = 1, 2, \ldots n),$$

we shall obtain (by the chief theorem of Lie's theory)

(9) $$x''_i = f_i (x_1, \ldots x_n, c_1, \ldots c_r)$$
$$(i = 1, 2, \ldots n),$$

where

(10) $$c_j = \varphi_j (a_1, \ldots a_r, b_1, \ldots b_r)$$
$$(j = 1, 2, \ldots r).$$

Whence it follows that $T_b T_a$ is also a transformation of the family, which, therefore, constitutes a group, — denoted in what follows by G.* The equivalence between the transformation T_c and the transformation resulting from the composition of T_a and T_b may be denoted by writing $T_c = T_b T_a$.

In general there is more than one system of functions $\varphi_1 (a, b)$, $\varphi_2 (a, b)$, etc., such that $T_c = T_b T_a$ if $c_j = \varphi_j (a, b)$ for $j = 1, 2, \ldots r$. For finite values of the a's and b's it may happen that every value of some one (or more) of the c's is infinite. In this case, from what precedes, it follows that, while both T_a and T_b are generated by infinitesimal transformations of the group, the transformation $T_b T_a$ resulting from their composition cannot be generated thus, and the group cannot properly be said to be continuous. But, if each of one (or more) of the systems of values of the c's is finite for the assigned values of the a's and b's, $T_b T_a$ can be generated by an infinitesimal transformation of the group.† A transformation of G which cannot be generated by an infinitesimal transformation of G may be termed *essentially singular.*

In what follows I shall modify the preceding notation, restricting the use of the symbols T_a, T_b, etc., to denote, unless otherwise stated, transformations of group G with finite parameters, and therefore generated, respectively, by the infinitesimal transformations $\sum_1^r a_j X_j$, $\sum_1^r b_j X_j$, etc.

* Lie: Transformationsgruppen, I. p. 158.

† Rettger: American Journal of Mathematics, XXII. p. 62.

§ 2.

The transformation obtained by the successive application to the manifold $x_1, x_2, \dots x_n$, in the order named, of the transformation $T_a^{-1} = T_{-a}$, inverse to T_a, and the transformation $T_{a+\delta a}$, where $\delta a_1, \delta a_2, \dots \delta a_r$ are infinitesimal (consequently, $T_{a+\delta a}$ is infinitely near to T_a), is one of the ∞^{r-1} of infinitesimal transformations of G. If we denote the parameters of this infinitesimal transformation by $\delta t\, b_j$ $(j = 1, 2, \dots r)$, δt being an infinitesimal constant, we have

(11) $$T_{\delta t b} = T_{a+\delta a}\, T_a^{-1} = T_{a+\delta a}\, T_{-a},$$

or

(11 a) $$T_{\delta t b}\, T_a = T_{a+\delta a}.$$

That is to say,

(11 b) $$f_i\,(x_1, \dots x_n, \delta t\, b_1, \dots \delta t\, b_r)$$
$$= f_i\,(f_1\,(x, -a), \dots f_n\,(x, -a), a_1 + \delta a_1, \dots a_r + \delta a_r)$$
$$(i = 1, 2, \dots n)$$

From this system of equations, which hold for all values of the x's, we derive, for the determination of b_1, b_2, etc., r equations independent of the x's and linear in $\delta a_1, \delta a_2$, etc., namely,

(12) $$\delta t\, b_j = A_{j1}\, \delta a_1 + A_{j2}\, \delta a_2 + \dots + A_{jr}\, \delta a_r$$
$$(j = 1, 2, \dots r),$$

where the A's are functions of $a_1, a_2, \dots a_r$. These equations written in Cayley's "abbreviated notation" are

(12 a) $$\delta t\,(b_1, b_2, \dots b_r) = \left(\begin{array}{cccc} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{array}\right\rvert \!\! \left(\delta a_1, \delta a_2, \dots \delta a_r\right)^*$$

* In this paper I employ the notation of Cayley's "Memoir on the Theory of Matrices," Philosophical Transactions, 1858, with the exception that the identical transformation will be denoted by I, whereas Cayley denotes this transformation by the symbol 1. In the notation and nomenclature invented by Cayley a linear substitution and a bilinear form is each represented by the square array of its coefficients, the matrix of the bilinear form or of the linear substitution. In accordance with Cayley's theory, if A denotes the matrix of the linear substitution

$$x'_i = \sum_1^n{}_\nu\, a_{i\nu}\, x_i \quad (i = 1, 2, \dots n),$$

and B the matrix of the linear substitution

Let ϕ_a denote the matrix of the bilinear form $-\sum_1^r{}_\mu \sum_1^r{}_\nu (\sum_1^r a_j c_{j\nu\mu}) y_\mu z_\nu$, namely,

$$\begin{pmatrix} -\Sigma a_j c_{j11}, & -\Sigma a_j c_{j21}, & \dots & -\Sigma a_j c_{jr1} \\ -\Sigma a_j c_{j12}, & -\Sigma a_j c_{j22}, & \dots & -\Sigma a_j c_{jr2} \\ \dots & \dots & \dots & \dots \\ -\Sigma a_j c_{j1r}, & -\Sigma a_j c_{j2r}, & \dots & -\Sigma a_j c_{jrr} \end{pmatrix}.$$

Let I denote the matrix unity (the identical transformation), and let e^{ϕ_a} denote the series $I + \phi_a + \frac{1}{2}\phi_a^2 + \dots$, which is convergent for any matrix ϕ_a. Then it will be found that

$$\begin{pmatrix} A_{11}, & A_{12}, & \dots & A_{1r} \\ A_{21}, & A_{22}, & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \end{pmatrix} = \frac{e^{\phi_a} - I}{\phi_a} = I + \tfrac{1}{2}\phi_a + \dots$$

Let now Δ_a denote the determinant of $\dfrac{e^{\phi} - I}{\phi_a}$. This determinant vanishes if and only if the a's are so chosen that

$$\begin{vmatrix} \Sigma a_j c_{j11} - 2k\pi\sqrt{-1}, & \Sigma a_j c_{j12} & , \dots \\ \Sigma a_j c_{j12} & , \Sigma a_j c_{j22} - 2k\pi\sqrt{-1}. & \\ \dots & \dots & \dots \end{vmatrix} = 0,$$

where k is some integer not zero. The values of the parameters a for which Δ_a vanishes may be termed *critical values* of the parameters. The critical values of the parameters a are, therefore, those values of the a's for which one or more of the roots of the characteristic equation of the matrix ϕ_a is an even multiple, not zero, of $\pi\sqrt{-1}$.

If $\Delta_a \neq 0$, we may take the b's arbitrarily, and then, from equations (12), derive expressions for $\delta a_1, \delta a_2, \dots \delta a_r$, as linear functions of $b_1, b_2, \dots b_r$. Thus, if $\Delta_a \neq 0$, we have

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$$x'_i = \sum_1^n{}_\nu b_{i\nu} x_i \ (i = 1, 2, \dots n),$$

$A \pm B$ denotes the matrix of the linear substitution

$$x'_i = \sum_1^n{}^\nu (a_{i\nu} \pm b_{i\nu}) x_i \ (i = 1, 2, \dots n),$$

and $A B$ the matrix of the linear substitution

$$x'_i = \sum_1^n{}^\mu \sum_1^n{}_\nu a_{i\mu} b_{\mu\nu} x_\nu \ (i = 1, 2, \dots n).$$

We shall then have $A(BC) = (AB)C$, $A(B \pm C) = AB + AC$, etc., but in general $AB \neq BA$.

$$(13)\quad (\delta a_1, \delta a_2, \ldots \delta a_r) = \frac{\delta t\, \phi_a}{e^{\phi_a} - I}(b_1, b_2, \ldots b_r)$$
$$= \delta t \begin{pmatrix} \frac{\mathrm{a}_{11}}{\Delta_a}, & \frac{\mathrm{a}_{12}}{\Delta_a}, & \ldots \\ \frac{\mathrm{a}_{21}}{\Delta_a}, & \frac{\mathrm{a}_{22}}{\Delta_a}, & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} (b_1, b_2, \ldots b_r),$$

or

$$(13a)\qquad \delta a_j = \frac{\delta t}{\Delta_a}(\mathrm{a}_{j1}\, b_1 + \mathrm{a}_{j2}\, b_2 + \ldots + \mathrm{a}_{jr}\, b_r)$$
$$(j = 1, 2, \ldots r),$$

where $\mathrm{a}_{\mu\nu}$ is the first minor of Δ_a relative to $A_{\nu\mu}$. The quantities δa_1, δa_2, δa_r, as determined by these equations, are infinitesimal if $\Delta_a \neq 0$, since then the constituents $\mathrm{a}_{\mu\nu}\, \Delta_a^{-1}$ of the matrix $\phi_a\,(e^{\phi_a} - I)^{-1}$ are finite. Therefore, if the parameters a are so chosen that $\Delta_a \neq 0$, we may take b_1, b_2, . . . b_r, arbitrarily, and, if δa_1, δa_2, . . . δa_r, are determined by equations (13), we have

$$T_{\delta t b}\; T_a = T_{a + \delta a},$$

where δa_1, δa_2, . . . δa_r are infinitesimal.

On the other hand, if the values assigned to the parameters a are critical values of the parameters, that is, if $\Delta_a = 0$, it will certainly in general, for arbitrary values of the b's, be impossible to determine infinitesimal increments δa_1, δa_2, . . . δa_r, of the parameters a to satisfy the symbolic equation

$$T_{\delta t b}\; T_a = T_{a + \delta a}.$$

In this case, it may, nevertheless, be possible to find a finite system of values c_1, c_2, . . . c_r, of the parameters such that $T_{\delta t b}\; T_a = T_c$; but group G may be such that, for at least special systems of values of the b's, no finite system c_1, c_2, . . . c_r, of the parameters can be found to satisfy this symbolic equation. E.g., let $r = 2$ and $X_1 = \dfrac{\partial}{\partial x_2}$, $X_2 = \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2}$. Then, if $a_2 = 2\pi\sqrt{-1}$, $\Delta_a = 0$; and if $b_1 \neq 0$, $b_2 = 0$, $T_a\, T_{tb}$ is essential singular for all values of $t \neq 0$.

From what precedes we have therefore the following theorem:

If T *is an arbitrary transformation of* G *for which* $\Delta_a \neq 0$, *the transformation* $\mathrm{T}_{tb}\,\mathrm{T}_a$, *the parameters* b_1, b_2, . . . b_r, *being arbitrary, can be generated by an infinitesimal transformation of the group, provided* t *is*

*sufficiently small.** *On the other hand if* $\Delta_a = 0$, *and the* b's *are properly chosen, the transformation* $T_{tb}\,T_a$ *may be essentially singular however small* t *may be taken.* Such a transformation T_a I term *non-essentially singular.*

If T_a' is non-essentially singular, that is, if $T_{tb}\,T_a$ is essentially singular however small t may be, a system of values $b'_1, b'_2, \ldots b'_r$, of the parameters can be found such that, however small t may be, $T_a\,T_{tb'}$ is essentially singular; and conversely.

§ 3.

Let $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_r$, and $b_1, b_2, \ldots b_r$, be any two systems of finite arbitrarily chosen values of the parameters of G, and let the transformation T_a be defined by the symbolic equation

$$T_a = T_{tb}\,T_{\bar{a}}, \tag{14}$$

where t is a variable quantity independent of the x's. We then have

$$a_j = \varphi_j(\bar{a}_1, \ldots \bar{a}_r, t b_1, \ldots t b_r) \qquad (j = 1, 2, \ldots r). \tag{15}$$

The differential equations satisfied by the a's are

$$\left(\frac{d a_1}{dt}, \frac{d a_2}{dt}, \ldots \frac{d a_r}{dt}\right) = \frac{\phi_a}{e^{\phi_a} - I}(b_1, b_2, \ldots b_r). \tag{16}$$

See *Bulletin of the American Mathematical Society* for February, 1900, p. 202.

Two groups G and $G^{(1)}$ are of the same *structure* (*Zusammensetzung*) if the structural constants (*Zusammensetzungconstanten*) c_{jkl} and $c^{(1)}_{jkl}$ are identical. For two groups of the same structure, the system of differential equations satisfied by the r dependent variables $a_j = \varphi_j(\bar{a}, t b)$ are the same. But the equations of the group may restrict the number of systems of the functions a_j, which differ in the initial values of the a_j's, in certain cases so that there shall be but one system of functions φ. Consequently, in the case of two groups of the same structure, one may contain essentially singular transformations and the other may contain no essentially singular transformation. Two such groups cannot properly be said to be isomorphic, since one is continuous and the other discontinuous.

* If $\Delta_a \neq 0$, $T_{tb}\,T_a$ may be essentially singular for an infinite number of values of t. But this assemblage of values of t has no derived assemblage.

It will be found that one or more of the roots of the characteristic equation of the matrix $e^{\phi_{tb}}e^{\phi_a}$ is equal to unity, irrespective of the value of t. If T_a is non-singular, and for every value of t each root of this equation is equal to unity, $T_{tb}T_a$ is non-singular for every value of t. Let it be assumed that T_a is non-singular, and that just $s < r$ of the roots of the characteristic equation of the matrix $e^{\phi_{tb}}e^{\phi_a}$ are equal to unity, irrespective of the value of t. Then the values of t for which $T_{a'} = T_{tb}\,T_a$ is singular (essentially or non-essentially) are included among those for which one, or more, of the remaining $r-s$ roots of this equation is equal to unity.

§ 4.

The infinitesimal transformation $\Sigma\, a_j\, X_j$ of group G, where the a's are quantities independent of the x's, is said to be *derived lineally* from the r independent infinitesimal transformations $X_1, X_2, \ldots X_r$ which generate G. The r infinitesimal transformations

$$a_1^{(k)}X_1 + a_2^{(k)}X_2 + \ldots + a_r^{(k)}X_r \; (k = 1, 2, \ldots r)$$

are independent if the determinant

$$\begin{vmatrix} a_j^{(k)} \\ (j, k = 1, 2, \ldots r) \end{vmatrix} \neq 0.$$

Any r independent infinitesimal transformation derived linearly from the X's also generates group G and may be substituted for the X's.*

Group G may contain an infinitesimal transformation $\Sigma\, a_j\, X_j$ commutative with each of the r infinitesimal transformations X_j which generate G, and, therefore, commutative with every infinitesimal transformation of G. Such a transformation Lie terms an *ausgezeichnete infinitesimale Transformation.*† In what follows it will be termed an *extraordinary infinitesimal transformation.*

Let G contain just s independent extraordinary infinitesimal transformation. In this case, from what has been said, we may suppose the X's so chosen that

$$X_j X_k = X_k X_j$$
$$(j = 1, 2, \ldots s \qquad k = 1, 2, \ldots r),$$

but that

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* Lie: Transformationsgruppen, I. p. 276.

† Lie: Continuinerliche Gruppen, p. 465.

$$X_j X_k \neq X_k X_j$$
$$(j, k = s+1, s+2, \ldots r)$$

We then have $c_{jkl} = 0$ for $j = 1, 2, \ldots s$, and $k, l = 1, 2, \ldots r$. And, as a consequence of the differential equations satisfied by the functions $a_j = \varphi_j (\bar{a}, t\, b)$, it will be found that

$$\frac{\partial\, \varphi_j (\bar{a}, b)}{\partial\, \bar{a}_k} \equiv 0, \qquad \frac{\partial\, \varphi_j (\bar{a}, b)}{\partial\, b_k} \equiv 0$$
$$(j = s+1, s+2, \ldots r \qquad k = 1, 2, \ldots s).$$

Moreover, we shall have

$$\varphi_j (\bar{a}, b) \equiv \bar{a}_j + b_j + \psi_j (\bar{a}_{s+1}, \ldots \bar{a}_r, b_{s+1} \ldots b_r)$$
$$(j = 1, 2, \ldots s).$$

From the differential equations satisfied by the functions $a_j = \varphi_j (\bar{a}, t\, b)$ it also follows that, if $c_{jkl} = 0$ for $j, k = 1, 2, \ldots r$, we then may put

$$\varphi_l (\bar{a}, b) = \bar{a}_l + b_l.\text{*}$$

§ 5.

If $r = 2$, group G either contains no extraordinary infinitesimal transformation or two linearly independent extraordinary infinitesimal transformations. In the first case, the infinitesimal transformation $\Sigma\, a_j X_j$ is commutative with no other infinitesimal transformation of G. In the second case, every two transformations of G are commutative.

If $r = 3$, and the structural constants are such that

$$\begin{vmatrix} c_{121}, & c_{131}, & c_{231} \\ c_{122}, & c_{132}, & c_{232} \\ c_{123}, & c_{133}, & c_{233} \end{vmatrix} \neq 0,\dagger$$

* This theorem, for the case in which G is a sub-group of the projective group, was given by Mr. Rettger in the American Journal of Mathematics, XXII. p. 73. As an example of this theorem let $X_1 = x_1 \dfrac{\partial}{\partial x_1}$, $X_2 = x_2 \dfrac{\partial}{\partial x_2}$, $X_3 = x_2 \dfrac{\partial}{\partial x_1}$.

Then $\qquad c_{jk1} = 0, c_{jk2} = 0 \quad (j, k = 1, 2, 3).$

And if

$T_b\, T_a = T_c$, $c_1 = a_1 + b_1 + 2\, k\, \pi \sqrt{-1}$, $c_2 = a_2 + b_2 + 2\, k'\, \pi \sqrt{-1}$, where k and k' are integers which may both be taken equal to zero.

† E. g., $X_1 = \dfrac{\partial}{\partial x_1}$, $X_2 = x_1 \dfrac{\partial}{\partial x_1}$, $X_3 = x_1^2 \dfrac{\partial}{\partial x_1}$.

the infinitesimal transformation $\Sigma a_j X_j$ is commutative with no other infinitesimal transformation of G. But, if $r = 3$ and this determinant vanishes, it is alway possible to find two distinct infinitesimal transformations $\Sigma a_j X_j$ and $\Sigma b_j X_j$ which shall be commutative.

Again, if $r > 3$, it is always possible to find two distinct infinitesimal transformations of G which shall be commutative.

The condition necessary and sufficient that two infinitesimal transformations $\Sigma a_j X_j$ and $\Sigma b_j X_j$ shall be commutative is that

(17) $$\phi_a (b_1, b_2, \ldots b_r) = 0;$$

or, what is the same thing, that

(18) $$\phi_b (a_1, a_2, \ldots a_r) = 0.$$

If $\Delta_a \neq 0$, the necessary and sufficient condition that every transformation of the group $G_1^{(b)}$ with a single parameter t shall be commutative with T_a, that is to say, that $T_a T_{tb} = T_{tb} T_a$ for every value of t, is that the infinitesimal transformations $\Sigma a_j X_j$ and $\Sigma b_j X_j$ shall be commutative.

In certain groups G, whatever the transformation T_a, provided $\Delta_a = 0$, it is always possible to find an infinitesmal transformation $\Sigma b_j X_j$, not commutative with $\Sigma a_j X_j$, which shall, nevertheless, generate a group $G_1^{(b)}$ with a single parameter t, every transformation T_{tb} of which shall be commutative with T_a. In other groups this is possible for certain transformations T_a for which $\Delta_a = 0$.

As an example of the former we have the group

$$X_1 = x_1 \frac{\partial}{\partial x_1}, \quad X_2 = x_2 \frac{\partial}{\partial x_2}, \quad X_3 = x_3 \frac{\partial}{\partial x_1}, \quad X_4 = x_3 \frac{\partial}{\partial x_2}$$

For this group $\Delta_a = 0$ if a_1 or a_2 is an even multiple, not zero, of $\pi \sqrt{-1}$. Let a_1, a_3, a_4 be arbitrary, and $a_2 = 2\pi\sqrt{-1}$, and let $a_1 b_3 - a_3 b_1 = 0$. Then $T_a T_{tb} = T_{tb} T_a$ for all values of t; but $\Sigma b_j X_j$ is not commutative with $\Sigma a_j X_j$ unless $a_2 b_4 - a_4 b_2 = 0$. If, however, $a'_1 = a_1, a'_2 = 0, a'_3 = a_3, a'_4 = 0, T_{a'} = T_a$ and $\Sigma b_j X_j$ is commutative $\Sigma a'_j X_j$.

If $\Delta_a = 0$, the necessary and sufficient condition that $T_a T_{tb} = T_{tb} T_a$ for all values of t is

(19) $$(e^{\phi_a} - I)(b_1, b_2, \ldots b_r) = 0.$$

It is to be noted that this condition is always satisfied if $\Sigma a_j X_j$ and $\Sigma b_j X_j$ are commutative. For then

$$\begin{aligned} \phi_a (b_1, b_2, \ldots b_r) &= 0 \\ \therefore \phi^2_a (b_1, b_2, \ldots b_r) &= 0 \\ \phi^3_a (b_1, b_2, \ldots b_r) &= 0 \\ \ldots\ldots\ldots\ldots \end{aligned}$$

Consequently, if $\Delta_a \neq 0$, and every transformation of the group $G_1^{(b)}$ is commutative with T_a, the above condition is satisfied.

§ 6.

Let

(20) $$T_a = T_{a'}.$$

Then every transformation of the sub-group $G_1^{(a')}$ is commutative with T_a. Therefore, if $\Delta_a \neq 0$, $\Sigma\, a'_j\, X_j$ is commutative with $\Sigma\, a_j\, X_j$. Whence it follows that

(21) $$T_{a-a'} = T_a\, T_{-a'} = T_a\, T_{a'}^{-1}$$

is the identical transformation.

If, however, $\Delta_a = 0$, it does not necessarily follow that $\Sigma\, a_j\, X_j$ and $\Sigma\, a'_j\, X_j$ are commutative; and therefore we do not necessarily have $T_{a-a'} = T_a\, T_{a'}^{-1}$.

E. g., let $r = 5$, and let

$$X_1 = x_1 \frac{\partial}{\partial x_1},\quad X_2 = x_2 \frac{\partial}{\partial x_2},\quad X_3 = x_3 \frac{\partial}{\partial x_1},\quad X_4 = x_3 \frac{\partial}{\partial x_2},\quad X_5 = x_4 \frac{\partial}{\partial x_4}.$$

Then $\Delta_a = 0$ if either a_1 or a_2 is an even multiple, not zero, of $\pi\sqrt{-1}$.

Let $$a_1 = 0,\ a_2 = 2\,k\,\pi\sqrt{-1} \neq 0,\ a_3 = 0,$$
$$a'_1 = 2\,k'\,\pi\sqrt{-1},\ a'_2 = 2\,k\,\pi\sqrt{-1} \neq 0,\ a'_5 = a_5,$$
where k and k' are integers. Then $\Delta_a = 0$, $\Delta_{a'} = 0$; and $T_a = T_{a'}$. But $\Sigma\, a_j\, X_j$ is not commutative with $\Sigma\, a'_j\, X_j$ unless $a'_4 = a_4$. Moreover $T_{a-a'}$ is not the identical transformation (i. e., $T_{a-a'} \neq T_a\, T_{a'}^{-1}$) unless $a'_4 = a_4$.

When $T_a = T_{a'}$ and $\Delta_a = 0$, it does not necessarily follow that $\Delta_{a'} = 0$. Thus, in the case of the group just considered, if

$$a_1 = 2\,k\,\pi\sqrt{-1} \neq 0,\ a_2 = 2\,k'\,\pi\sqrt{-1} \neq 0,$$
$$a'_1 = 0,\ a'_2 = 0,\ a'_3 = 0,\ a'_4 = 0,\ a'_5 = a_5,$$

we have $T_a = T_{a'}$, and $\Delta_a = 0$; but $\Delta_{a'} \neq 0$.

§ 7.

The equations of the general infinitesimal transformations of the adjoined group Γ of group G are in Cayley's matrical notation

(22) $$(a'_1,\ a'_2,\ \ldots\ a'_r) = (I + \delta t\, \phi_a \between a_1,\ a_2,\ \ldots\ a_r).$$

The successive application of this infinitesimal transformation to the manifold $a_1, a_2, \ldots a_r$ gives the general transformation of Γ, namely,

(23) $$(a_1^{(1)},\ a_2^{(1)},\ \ldots\ a_r^{(1)}) = (e^{\phi_a} \between a_1,\ a_2,\ \ldots\ a_r).$$

If

$$(24)\qquad (a_1^{(2)}, a_2^{(2)}, \ldots a_r^{(2)}) = (e^{\phi_\beta}\between a_1^{(1)}, a_2^{(1)}, \ldots a_r^{(1)}),$$

we have

$$(25)\qquad (a_1^{(2)}, a_2^{(2)}, \ldots a_r^{(2)}) = (e^{\phi_\beta}\, e^{\phi_a}\between a_1, a_2, \ldots a_r).$$

Thus the general transformation of the adjoined may be represented by the matrix e^{ϕ_a}; and the result of the two successive transformations e^{ϕ_a} and e^{ϕ_β} is represented by the matrix $e^{\phi_\beta}\, e^{\phi_a}$ obtained by their composition.

If $\gamma_1, \gamma_2, \ldots \gamma_r$ are so chosen that

$$(26)\qquad T_\gamma = T_\beta\, T_a,$$

that is, if

$$(27)\qquad \gamma_j = \varphi_j\,(a_1, \ldots a_r, \beta_1, \ldots \beta_r)\\ (j = 1, 2, \ldots r),$$

then

$$(28)\qquad e^{\phi_\gamma} = e^{\phi_\beta}\, e^{\phi_a}.$$

If G contains no extraordinary infinitesimal transformation, Γ has the same structure as G and the r parameters a are all essential.* In this case if we put

$$(29)\qquad e^{\phi_a} = e^{\phi_{t\beta}}\, e^{\phi_{\bar{a}}},$$

the a's, as functions of t, are defined by the differential equations of p. 584; thus we have

$$(30)\qquad \left(\frac{d\,a_1}{d\,t}, \frac{d\,a_2}{d\,t}, \ldots \frac{d\,a_r}{d\,t}\right) = \frac{\phi_a}{e^{\phi_a} - I}(\beta_1\, \beta_2, \ldots \beta_r).$$

It will be found, however, that the symbolic equation (29), in general, defines more systems of functions a than the symbolic equation

$$T_a = T_{t\beta}\, T_{\bar{a}}.$$

E.g., let $r = 2$, and

$$X_1 = \frac{\partial}{\partial x_2},\quad X_2 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}.$$

Then, if

$$T_c = T_b\, T_a,$$

$$c_1 = \frac{a_2 + b_2}{e^{a_2 + b_2} - 1}\left(a_1\, e^{b_2}\frac{e^{a_2} - 1}{a_2} + b_1\frac{e^{b_2} - 1}{b_2}\right),$$

* Lie: Transformationsgruppen, I. p. 277.

$$c_2 = a_2 + b_2.$$

For this group

$$\phi_a = \begin{pmatrix} a_2, & a_1 \\ 0, & 0 \end{pmatrix};$$

and, therefore, the equations of Γ are

$$(a'_1, a'_2) = e^{\begin{pmatrix} a_2, & a_1 \\ 0, & 0 \end{pmatrix}} (a_1, a_2) = \begin{pmatrix} e^{a_2}, & a_1 \dfrac{e^{a_2} - 1}{a_2} \\ 0, & 0 \end{pmatrix} (a_1, a_2).$$

Therefore, if

$$e^{\phi_\gamma} = e^{\phi_\beta} e^{\phi_a},$$

we have

$$\gamma_1 = \frac{a_2 + \beta_2 + 2 k \pi \sqrt{-1}}{e^{a_2 + \beta_2} - 1} \left(a_1 e^{\beta_2} \frac{e^{a_2} - 1}{a_2} + \beta_1 \frac{e^{\beta_2} - 1}{\beta_2} \right),$$

$$\gamma_2 = a_2 + \beta_2 + 2 k \pi \sqrt{-1},$$

where k is any integer.

If G contains one or more extraordinary infinitesimal transformations, and $\gamma_1, \gamma_2, \ldots \gamma_r$ are determined by equations (27), we shall still have $e^{\phi_\gamma} = e^{\phi_\beta} e^{\phi_a}$; but these conditions, though sufficient, are not all necessary. If G contains just s independent extraordinary infinitesimal transformations, just $r - s$ of the parameters a are essential.

If the s independent extraordinary infinitesimal transformations of G are $X_1, X_2 \ldots X_s$, then $a_1, a_2, \ldots a_s$ do not appear in ϕ_a; and if

$$e^{\phi_a} = e^{\phi_{t\beta}} e^{\phi_{\bar{a}}},$$

we have for the determination of the remaining parameters a the differential equations

$$\left(\frac{d a_{s+1}}{dt}, \ldots \frac{d a_r}{dt} \right) = \begin{pmatrix} \dfrac{\mathfrak{a}^{(a)}_{s+1, s+1}}{\Delta_a}, & \ldots & \dfrac{\mathfrak{a}^{(a)}_{s+1, r}}{\Delta_a} \\ \dfrac{\mathfrak{a}^{(a)}_{s+2, s+1}}{\Delta}, & \ldots & \dfrac{\mathfrak{a}^{(a)}_{s+2, r}}{\Delta_a} \\ \cdots & \cdots & \cdots \end{pmatrix} (b_{s+2}, \ldots b_r),$$

where $\dfrac{\mathfrak{a}^{(a)}_{\mu, \nu}}{\Delta_a}$ denotes the result of substituting $a_{s+1}, \ldots a_r$ for $a_{s+1}, \ldots a_r$, respectively in the functions $\dfrac{\mathfrak{a}_{\mu\nu}}{\Delta_a}$ of p. 583.

If the group G is continuous, the adjoined group Γ is continuous. Therefore, if Γ is discontinuous, group G and every group of the same structure is discontinuous.

§ 8.

For certain types of structure every root of the characteristic equation of the matrix ϕ_a is zero irrespective of the values of the parameters a. In this case $\Delta_a \equiv 1$, and the group is continuous.

For two groups of the same structure, but such that $\Delta_a \not\equiv 1$, one may be continuous and the other discontinuous. As remarked above, if the adjoined is discontinuous, group G and every group of the same structure is discontinuous.

I give below a table exhibiting the result of an investigation by my pupil, Mr. S. E. Slocum, of the continuity of all types of groups with either two, three, or four parameters. It will be seen that for every type of structure for which $\Delta_a \not\equiv 1$, there is at least one group which is discontinuous. Mr. Slocum has found that every real group is continuous which possesses any one of the several structures distinguished in the table by an asterisk; and that the real group which possesses the structure marked in the table thus (†) is discontinuous. Also that every real group of the structure $(X_1, X_2) = 0, (X_1, X_3) = - X_2, (X_2, X_3) = - X_1$, is discontinuous. For this group

$$\Delta_a = \frac{(e^{a_3\sqrt{-1}} - 1)(e^{-a_3\sqrt{-1}} - 1)}{a^2_3}$$

[In what follows (X_j, X_k) denotes $X_j X_k - X_k X_j$.]

Groups with two Parameters ($r = 2$).

Type I. $(X_1, X_2) \equiv X_1$.* $\Delta_a = \dfrac{e^{a_2} - 1}{a_2}$.

Adjoined group continuous.

Parameter group discontinuous; also group

$$\frac{\partial}{\partial x_2}, \quad x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}.$$

Type II. $(X_1, X_2) \equiv 0$.* $\Delta_a = 1$.

All groups of this type are continuous.

Groups with three Parameters ($r = 3$).

Type I. $(X_1, X_2) \equiv X_1, (X_1, X_3) \equiv 2 X_2, (X_2, X_3) \equiv X_3$.†

$$\Delta_a = \frac{e^{\sqrt{a_2^2 - 4 a_1 a_3}} - 1}{\sqrt{a_2^2 - 4 a_1 a_3}} \quad \frac{e^{-\sqrt{a_2^2 - 4 a_1 a_3}} - 1}{-\sqrt{a_2^2 - 4 a_1 a_3}}$$

Adjoined group continuous.

Group $x_1 \dfrac{\partial}{\partial x_2}$, $x_1 \dfrac{\partial}{\partial x_1} - x_2 \dfrac{\partial}{\partial x_2}$, $x_2 \dfrac{\partial}{\partial x_1}$, discontinuous.

Type II. $(X_1, X_2) \equiv 0, (X_1, X_3) \equiv X_1, (X_2, X_3) \equiv \beta X_2 \, (\beta \neq 0, 1)$.*

$$\Delta_a = \frac{(e^{a_3} - 1)\,(e^{a_3 \beta} - 1)}{a_3^2 \beta}.$$

Adjoined group $-a_3 \dfrac{\partial}{\partial a_1}, -a_3 \beta \dfrac{\partial}{\partial a_2}, \; a_1 \dfrac{\partial}{\partial a_1} + a_2 \beta \dfrac{\partial}{\partial a_2}$,

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$\frac{\partial}{\partial x_1}, \; x_1 \frac{\partial}{\partial x_1} + \beta x_2 \frac{\partial}{\partial x_2} + a \frac{\partial}{\partial x_3}, \; \frac{\partial}{\partial x_2}.$$

Type III. $(X_1, X_2) \equiv 0, (X_1, X_3) \equiv X_1, (X_2, X_3) \equiv X_2$.*

$$\Delta_a = \left(\frac{e^{a_3} - 1}{a_3}\right)^2.$$

Adjoined group continuous.
Parameter group discontinuous; also group

$$x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \; \frac{\partial}{\partial x_1}, \; \frac{\partial}{\partial x_2}.$$

Type IV. $(X_1, X_2) \equiv 0, \; (X_1, X_3) \equiv X_1, \; (X_2, X_3) \equiv X_1 + X_2$.*

$$\Delta_a = \left(\frac{e^{a_3} - 1}{a_3}\right)^2.$$

Adjoined group

$$-a_3 \frac{\partial}{\partial a_1}, -a_3 \frac{\partial}{\partial a_1} - a_3 \frac{\partial}{\partial a_2}, (a_1 + a_2) \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2},$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$2 x_3 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}, \; \frac{\partial}{\partial x_1}, \; \frac{\partial}{\partial x_3}.$$

Type V. $(X_1, X_2) \equiv 0, \; (X_1, X_3) \equiv X_1, \; (X_2, X_3) \equiv 0$.*

$$\Delta_a = \frac{e^{a_3} - 1}{a_3}.$$

Adjoined group continuous.

Parameter group discontinuous; also group

$$\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}.$$

Type VI. $(X_1, X_2) \equiv 0, (X_1, X_3) \equiv 0, (X_2, X_3) \equiv X_1.$*

$$\Delta_a = 1.$$

All groups of this type are continuous.

Type VII. $(X_1, X_2) \equiv 0, (X_1, X_3) \equiv 0, (X_2, X_3) \equiv 0.$*

$$\Delta_a = 1.$$

All groups of this type are continuous.

Groups with four Parameters ($r = 4$).

A. Without three-parameter involution group.

Type I. $\begin{cases} (X_1, X_2) \equiv X_1, (X_1, X_3) \equiv 2\, X_2, (X_2, X_3) \equiv X_3, \\ (X_1, X_4) \equiv 0, \ (X_2, X_4) \equiv 0, (X_3, X_4) \equiv 0. \end{cases}$

$$\Delta_a = \frac{e^{\sqrt{a_2^2 - 4\, a_1 a_3}} - 1}{\sqrt{a_2^2 - 4\, a_1 a_3}} \cdot \frac{e^{-\sqrt{a_2^2 - 4\, a_1 a_3}} - 1}{-\sqrt{a_2^2 - 4\, a_1 a_3}}.$$

Adjoined group continuous.

Group $x_1 \frac{\partial}{\partial x_2}, \quad x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}, \quad x_2 \frac{\partial}{\partial x_1}, \quad x_3 \frac{\partial}{\partial x_3}$, discontinuous.

Type II. $\begin{cases} (X_1, X_2) \equiv 0, (X_1, X_3) \equiv 0, (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv \beta\, X_1, (X_2, X_4) \equiv X_2, (X_3, X_4) \\ \qquad \equiv (\beta - 1)\, X_3 \ (\beta \neq 1).^* \end{cases}$

$$\Delta_a = \left(\frac{e^{a_4} - 1}{a_4}\right) \left(\frac{e^{a_4 \beta} - 1}{a_4 \beta}\right) \left(\frac{e^{a_4 (\beta - 1)} - 1}{a_4 (\beta - 1)}\right).$$

Adjoined group

$$- a_4 \beta \frac{\partial}{\partial a_1}, - a_3 \frac{\partial}{\partial a_1} - a_4 \frac{\partial}{\partial a_2}, \quad a_2 \frac{\partial}{\partial a_1} - a_4 (\beta - 1) \frac{\partial}{\partial a_3}, \quad a_1 \beta \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + a_3 (\beta - 1) \frac{\partial}{\partial a_3}$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_1}, \quad x_1 \frac{\partial}{\partial x_2}, \quad x_1 \frac{\partial}{\partial x_1} + \beta\, x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Type III. $\begin{cases} (X_1, X_2) \equiv 0, (X_1, X_3) \equiv 0, (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv 2\,X_1, (X_2, X_4) \equiv X_2, (X_3, X_4) \equiv 2\,X_2 + X_3.\text{*} \end{cases}$

$$\Delta_a = \left(\frac{e^{a_4} - 1}{a_4}\right)^2 \left(\frac{e^{2a_4} - 1}{2\,a_4}\right).$$

Adjoined group

$$-2\,a_4 \frac{\partial}{\partial a_1}, \; -a_3 \frac{\partial}{\partial a_1} - a_4 \frac{\partial}{\partial a_2}, \quad a_2 \frac{\partial}{\partial a_1} - 2\,a_4 \frac{\partial}{\partial a_2} - a_4 \frac{\partial}{\partial a_3},$$

$$2\,a_1 \frac{\partial}{\partial a_1} + (a_2 + 2\,a_3) \frac{\partial}{\partial a_2} + a_3 \frac{\partial}{\partial a_3},$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$2\,x_3 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \quad 2\,x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_3}.$$

Type IV. $\begin{cases} (X_1, X_2) \equiv 0, (X_1, X_3) \equiv 0, (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv X_1, (X_2, X_4) \equiv X_2, (X_3, X_4) \equiv 0.\text{*} \end{cases}$

$$\Delta_a = \left(\frac{e^{a_4} - 1}{a_4}\right)^2.$$

Adjoined group continuous.

Group $\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_1}, \quad x_1 \frac{\partial}{\partial x_2}, \quad x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$, discontinuous.

Type V. $\begin{cases} (X_1, X_2) \equiv 0, (X_1, X_3) \equiv 0, (X_2, X_3) \equiv X_2, \\ (X_1, X_4) \equiv X_1, (X_2, X_4) \equiv 0, (X_3, X_4) \equiv 0.\text{*} \end{cases}$

$$\Delta_a = \left(\frac{e^{a_4} - 1}{a_4}\right) \left(\frac{e^{a_3} - 1}{a_3}\right).$$

Adjoined group continuous.
Parameter group discontinuous; also group

$$\frac{\partial}{\partial x_1}, \quad x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_2}, \quad x_2 \frac{\partial}{\partial x_2}.$$

B. With three-parameter involution group.

Type I. $\begin{cases} (X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0, \\ (X_1, X_4) \equiv \alpha\,X_1, (X_2, X_4) \equiv \beta\,X_2, (X_3, X_4) \equiv \gamma\,X_3\text{*} \end{cases}$

$$(\alpha \mp \beta \mp \gamma).$$

$$\Delta_a = \left(\frac{e^{a_4\alpha} - 1}{a_4\,\alpha}\right) \left(\frac{e^{a_4\beta} - 1}{a_4\,\beta}\right) \left(\frac{e^{a_4\gamma} - 1}{a_4\,\gamma}\right).$$

Adjoined group

$$-a_4\alpha\frac{\partial}{\partial a_1},\ -a_4\beta\frac{\partial}{\partial a_2},\ -a_4\gamma\frac{\partial}{\partial a_3},\quad a_1\alpha\frac{\partial}{\partial a_1}+a_2\beta\frac{\partial}{\partial a_2}+a_3\gamma\frac{\partial}{\partial a_3}$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3},\quad \alpha x_1\frac{\partial}{\partial x_1}+\beta x_2\frac{\partial}{\partial x_2}+\gamma x_3\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}$$

Type II. $\begin{cases}(X_1, X_2)\equiv(X_2, X_3)\equiv(X_3, X_1)\equiv 0,\\ (X_1, X_4)\equiv\alpha X_1,\ (X_2, X_4)\equiv\beta X_2,\ (X_3, X_4)\end{cases}$

$$\equiv X_2+\beta X_3.\text{*}$$

$(\alpha \neq \beta)$.

$$\Delta_a=\left(\frac{e^{a_4\alpha}-1}{a_4\alpha}\right)\left(\frac{e^{a_4\beta}-1}{a_4\beta}\right)^2.$$

Adjoined group

$$-a_4\alpha\frac{\partial}{\partial a_1},\ -a_4\beta\frac{\partial}{\partial a_2},\ -a_4\frac{\partial}{\partial a_2}-a_4\beta\frac{\partial}{\partial a_3},\quad a_1\alpha\frac{\partial}{\partial a_1}$$
$$+(a_2\beta+a_3)\frac{\partial}{\partial a_2}+a_3\beta\frac{\partial}{\partial a_3}$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3},\quad x_3\frac{\partial}{\partial x_2}+\alpha x_1\frac{\partial}{\partial x_1}+\beta\left(x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}\right)+\frac{\partial}{\partial x_4},$$

Type III. $\begin{cases}(X_1, X_2)\equiv(X_2, X_3)\equiv(X_3, X_1)\equiv 0,\\ (X_1, X_4)\equiv X_1,\ (X_2, X_4)\equiv X_1+X_2,\ (X_3, X_4)\end{cases}$

$$\equiv X_2+X_3.$$

$$\Delta_a=\left(\frac{e^{a_4}-1}{a_4}\right)^3.$$

Adjoined group

$$-a_4\frac{\partial}{\partial a_1},\ -a_4\frac{\partial}{\partial a_1}-a_4\frac{\partial}{\partial a_2},\ -a_4\frac{\partial}{\partial a_2}-a_4\frac{\partial}{\partial a_3},\quad (a_1+a_2)\frac{\partial}{\partial a_1}$$
$$+(a_2+a_3)\frac{\partial}{\partial a_2}+a_3\frac{\partial}{\partial a_3}$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$2x_3\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_3}+x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3},\quad \frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3}.$$

Type III.′ $\begin{cases}(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,\\(X_1, X_4) \equiv 0,\ (X_2, X_4) \equiv X_1,\ (X_3, X_4) \equiv X_2.^*\end{cases}$

$$\Delta_a = 1.$$

All groups of this type are continuous.

Type IV. $\begin{cases}(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3\ X_1) \equiv 0,\\(X_1, X_4) \equiv a\, X_1,\ (X_2, X_4) \equiv a\, X_2,\ (X_3, X_4) \equiv \gamma\, X_3.^*\end{cases}$

$$(a \neq \gamma)$$

$$\Delta_a = \left(\frac{e^{a_4 a} - 1}{a_4\, a}\right)^2 \left(\frac{e^{a_4 \gamma} - 1}{a_4\, \gamma}\right).$$

Adjoined group

$$-a_4\, a \frac{\partial}{\partial a_1},\ -a_4\, a \frac{\partial}{\partial a_2},\ -a_4\, \gamma \frac{\partial}{\partial a_3},\quad a_1\, a \frac{\partial}{\partial a_1} + a_2\, a \frac{\partial}{\partial a_2} + a_3\, \gamma \frac{\partial}{\partial a_3}$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3},\quad a\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) + \gamma\, x_3 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}.$$

Type V. $\begin{cases}(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,\\(X_1, X_4) \equiv X_1,\ (X_2, X_4) \equiv X_2,\ (X_3, X_4) \equiv X_2 + X_3.^*\end{cases}$

$$\Delta_a = \left(\frac{e^{a_4} - 1}{a_4}\right)^3.$$

Adjoined group

$$-a_4 \frac{\partial}{\partial a_1},\ -a_4 \frac{\partial}{\partial a_2},\ -a_4 \frac{\partial}{\partial a_2} - a_4 \frac{\partial}{\partial a_3},\quad a_1 \frac{\partial}{\partial a_1} + (a_2 + a_3) \frac{\partial}{\partial a_2} + a_3 \frac{\partial}{\partial a_3}$$

discontinuous. Therefore, all groups of this type are discontinuous; e. g., group

$$\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3},\quad x_3 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

Type V.′ $\begin{cases}(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,\\(X_1, X_4) \equiv 0,\ (X_2, X_4) \equiv 0,\ (X_3, X_4) \equiv X_2.^*\end{cases}$

$$\Delta_a = 1.$$

All groups of this type are continuous.

Type VI. $\begin{cases}(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,\\(X_1, X_4) \equiv X_1,\ (X_2, X_4) \equiv X_2,\ (X_3, X_4) \equiv X_3.^*\end{cases}$

$$\Delta_a = \left(\frac{e^{a_4} - 1}{a_4}\right)^3.$$

Adjoined group continuous.

Parameter group discontinuous; also group

$$\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}, \quad x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}.$$

$$\text{Type VI.}' \left\{ \begin{array}{l} (X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0, \\ (X_1, X_4) \equiv 0, (X_2, X_4) \equiv 0, (X_3, X_4) \equiv 0. \end{array} \right.$$

$$\Delta_a = 1.$$

All groups of this type are continuous.

P. S. Since what precedes was written, I have found that the infinitesimal transformations of the parameter group, given in equations (13), § 2 (from which follow the differential equations (16), § 3), had already been obtained by Schur and Engel; and that to Engel is due the determination of the values of the a's for which $\Delta_a = 0$. See Lie: *Transformationsgruppen*, III, pp. 760, 794.

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